Chapter 6 The Laplace Transform

6.1 Definition of the Laplace Transform

Recall Improper integral

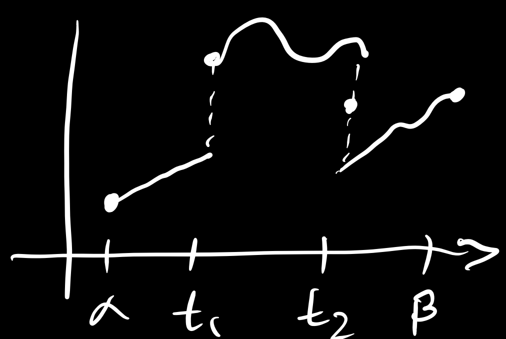
$$\int_a^\infty f(t)\,dt = \lim_{b\to\infty}\int_a^b f(t)\,dt$$

If $\int_a^b f(t)\,dt$ exists for all $b > a$, and the limit as $b\to\infty$ exists, then the improper integral is said to converge. Otherwise, it diverges.

Remark Showing convergence directly may be difficult, so we use comparison tests.

Def A function f is called piecewise continuous on an interval $\alpha \le t \le \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that

1. f is continuous on each subinterval $t_{i-1} < t < t_i$
2. f approaches a finite limit as each endpoint of each subinterval is approached from within the subinterval.



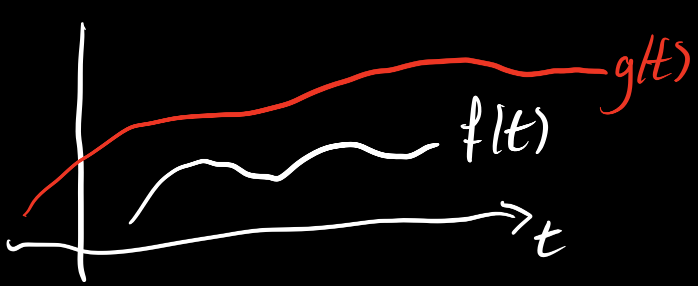
<u>Remark</u> 1. We may now write $\int_\alpha^\beta f(t)\,dt$ as

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \int_{t_2}^{\beta} f(t)\,dt$$

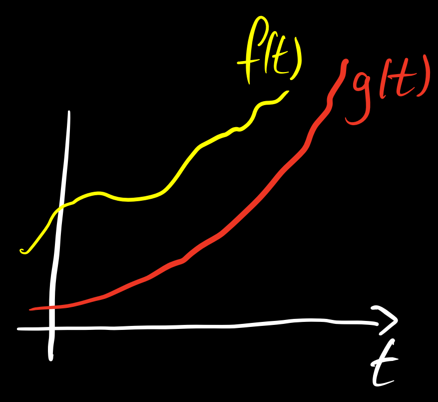
Thm If is piecewise continuous for $a \le t$, if $|f(t)| \le g(t)$ when $t \ge M$ for some positive constant, and if $\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ converges.

On the other hand, if $f(t) \ge g(t) \ge 0$ for $t \ge M$ and $\int_M^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ diverges.

Converges



diverges



The Laplace Transform

Integral Transform

$$F(s) = \int_{\alpha}^{\beta} K(s,t)\, f(t)\, dt$$

- $K(s,t)$ is given and called the kernel of the transformation
- $F(s)$ is called the transform of $f(t)$

Def Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt$$

Here, the kernel is $K(s,t) = e^{-st}$

Steps for using the Laplace transform to solve a DE:

1. Transform the DE from t-domain to a simpler problem in the s-domain
2. Solve the algebraic problem to find $F(s)$
3. Recover f from it transform F

Thm Suppose that

(i) f is piecewise continuous on $0 \le t \le b$ for any positive b

(ii) There are constants K, a, and M so that

$$|f(t)| \le K e^{at} \text{ when } t \ge M.$$

Then $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$.

Proof The Laplace Transform of f is

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^M f(t) e^{-st} dt + \int_M^\infty f(t) e^{-st} dt$$

Now, $\int_0^M f(t) e^{-st} dt < \infty$ by (i)

and

$$\left| \int_M^\infty f(t) e^{-st} dt \right| \le \int_M^\infty |f(t)| e^{-st} dt$$

$$\le \int_M^\infty K e^{at} e^{-st} dt \quad \text{by (ii)}$$

$$= \int_M^\infty K e^{(a-s)t} dt$$

$$= \frac{K e^{(a-s)t}}{a-s} \Big|_M^\infty$$

$$= \lim_{t \to \infty} \frac{K e^{(a-s)t}}{a-s} - \frac{K e^{(a-s)M}}{a-s}$$

$$= \frac{K e^{(a-s)M}}{s-a} + \lim_{t \to \infty} \frac{K e^{(a-s)t}}{a-s}$$

$< \infty$

when $a < s$.

Some Laplace Transforms

1. $f(t) = c$, c is some constant

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty c\, e^{-st}\, dt$$

$$= \left. \frac{c\, e^{-st}}{-s} \right|_0^\infty$$

$$= 0 - \frac{c}{-s}$$

$$= \frac{c}{s}$$

2. $f(t) = e^{at}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{at} e^{-st}\, dt$$

$$= \int_0^\infty e^{(a-s)t}\, dt$$

$$= \frac{e^{(a-s)t}}{a-s}\Big|_0^\infty$$

$$= 0 - \frac{1}{a-s}$$

$$= \frac{1}{s-a} \quad \text{when } a < s$$

3. $f(t) = \sin(at)$

$$\mathcal{L}\{f(t)\} = \int_0^\infty \sin(at)\, e^{-st}\, dt$$

$u = \sin(at)$ $\quad dv = e^{-st} dt$

$du = a\cos(at)dt$ $\quad v = \frac{e^{-st}}{-s}$

$$= \frac{e^{-st}\sin(at)}{-s}\Big|_0^\infty - \int_0^\infty \frac{a}{-s} e^{-st} \cos(at)\, dt$$

$$= 0 - 0 + \int_0^\infty \frac{a}{s} e^{-st} \cos(at)\, dt$$

$$= \frac{a}{s} \int_0^\infty e^{-st} \cos(at)\, dt$$

$u = \cos(at)$ $dv = e^{-st} dt$

$du = -a\sin(at) dt$ $v = \frac{e^{-st}}{-s}$

$$= \frac{a}{s}\left(\frac{e^{-st}\cos(at)}{-s}\Big|_0^\infty - \int_0^\infty \frac{a}{s} e^{-st} \sin(at)\, dt \right)$$

$$= \frac{a}{s}\left(0 - \frac{1}{-s} - \frac{a}{s}\int_0^\infty e^{-st} \sin(at)\, dt \right)$$

Now, $F(s) = \mathcal{L}\{f(t)\}$ so

$$F(s) = \frac{a}{s}\left(\frac{1}{s} - \frac{a}{s}F(s)\right)$$

$$F(s) + \frac{a^2}{s^2}F(s) = \frac{a}{s^2}$$

$$F(s) = \frac{a}{s^2} \cdot \frac{1}{1 + \frac{a^2}{s^2}}$$

$$= \frac{a}{s^2+a^2}$$

Thus,

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$$

4. $f(t) = \cos(at)$ (Similar process as 3.)

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+a^2}$$

Remark Laplace Transform is a linear operator

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty (c_1 f_1(t) + c_2 f_2(t)) e^{-st}\, dt$$

$$= \int_0^\infty c_1 f_1(t) e^{-st} + c_2 f_2(t) e^{-st}\, dt$$

$$= c_1 \int_0^\infty f_1(t) e^{-st} dt + c_2 \int_0^\infty f_2(t)\, dt$$

$$= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$$

$$= c_1 F_1(s) + c_2 F_2(s)$$

Ex Find the Laplace transform of

$$f(t) = 5e^{-2t} - 3\sin(4t)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{5e^{-2t} - 3\sin(4t)\}$$

$$= 5\mathcal{L}\{e^{-2t}\} - 3\mathcal{L}\{\sin(4t)\}$$

$$= 5 \cdot \frac{1}{s-(-2)} - 3\frac{4}{s^2+4^2}$$

$$= \frac{5}{s+2} - \frac{12}{s^2+16}$$

§6.2 Solution of Initial Value Problems.

We will now see how the Laplace Transform can be used to solve initial value problems.

Thm Suppose f is continuous and f' is piecewise continuous on $0 \le t \le b$. Suppose that there are constants K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists and

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

Proof Consider the integral

$$\int_0^b e^{-st} f'(t)\,dt = \int_{t_0}^{t_1} f'(t) e^{-st}\,dt + \int_{t_1}^{t_2} f'(t) e^{-st}\,dt$$

$$+ \cdots + \int_{t_{n-1}}^{t_n} f'(t) e^{-st}\,dt$$

$$= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f'(t) e^{-st}\,dt$$

Since $f'(t)$ is piecewise continuous. Now,

$$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f'(t)\, e^{-st}\, dt \qquad u = e^{-st} \quad dv = f'(t)\,dt$$

$$du = -se^{-st}\,dt \quad v = f(t)$$

$$= \sum_{k=0}^{n-1} f(t)\,e^{-st}\Big|_{t_k}^{t_{k+1}} + \int_{t_k}^{t_{k+1}} se^{-st} f(t)\,dt$$

$$= \cancel{f(t_1)\,e^{-st_1}} - f(t_0)\,e^{-st_0} \cancel{+ f(t_2)\,e^{-st_2}} \cancel{- f(t_1)\,e^{-st_1}}$$

$$\cancel{+ \cdots} + f(t_n)\,e^{-st_n} - \cancel{f(t_{n-1})\,e^{-st_{n-1}}}$$

$$+ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} s\,e^{-st} f(t)\,dt$$

$$= f(t_n)\,e^{-st_n} - f(t_0)\,e^{-st_0} + s \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} e^{-st} f(t)\,dt$$

$$= f(b)\,e^{-sb} - f(0) + s\int_0^b e^{-st} f(t)\,dt$$

$$\longrightarrow s\int_0^\infty e^{-st} f(t)\,dt - f(0)$$

as $b \to \infty$.

Therefore, $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$.

Remark We can now use this result to find $\mathcal{L}\{f''(t)\}$.

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$
$$= s\left(s\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$
$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

For $\mathcal{L}\{f^{(n)}(t)\}$, we have the following Corollary.

Cor $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0)$
$- \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$

Ex $y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$

First, we transform to the s-domain.

Let $Y(s) = \mathcal{L}\{y(t)\}$. Then

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$$

Now,

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$
$$= s^2 Y(s) - s \cdot 1 - 0$$
$$= s^2 Y(s) - s$$

and

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$
$$= sY(s) - 1$$

Then, we have

$$s^2 Y(s) - s - (sY(s) - 1) - 2Y(s) = 0$$

Now, we solve for $Y(s)$,

$$s^2 Y(s) - s - sY(s) + 1 - 2Y(s) = 0$$

$$Y(s)\left(s^2 - s - 2\right) = s - 1$$

$$Y(s) = \frac{s-1}{s^2 - s - 2}$$

We have found the solution in the s-domain. The final step is to go back to the t-domain.

<u>Remark</u> Notice that our differential equation became an algebraic equation. This vastly simplifies the problem, but we are now left with returning to the t-domain.

Inverting the Laplace Transform

We will not explicitly use the inverse Laplace Transform, but rather compare the solution $Y(s)$ to known transformed functions to infer the inversion. Again, the key idea is linearity.

Suppose

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

and $f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}, \ldots, f_n(t) = \mathcal{L}^{-1}\{F_n(s)\}$

then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{F_1(s)\} + \cdots + \mathcal{L}^{-1}\{F_n(s)\}.$$